# Solitons in field theory.

Cora Dvorkin
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# 1 Introduction: Solitons in field theory.

The classical finite-energy solutions of field theory are generally called *solitons*. Most of the simple field theories which we are familiar have the property that all of their non-singular solutions of finite total energy are dissipative. This is the case of Maxwell equation, the Klein-Gordon equation, etc [3].

However, there are some field theories that posses non-dissipative solutions of finite energy. Among these are some spontaneously broken gauge theories. The most simple case are time-independent solutions, lumps of energy holding themselves together by their own self-interaction.

In order to explain this, we will take as an example a $\lambda\phi^4$ theory in $1+1$ dimension. The Lagrangian can be written as

$$\mathcal{L} = \int [\frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - V(\phi)]dx, \tag{1}$$

where

$$V(\phi) = \frac{\lambda}{2}(\phi^2 - a^2)^2 \tag{2}$$

The vacuum expectation value of the field is

$$\phi = \pm a = \pm\sqrt{\frac{\mu^2}{\lambda}},$$

and the ground state energy is $E = 0$. Here, we are studying small oscillations around the vacuum expectation value of the field ($\phi = \mathrm{v} + \chi$), and $\mu$ is the mass of the meson $\chi$.

There exists a static, finite energy solution to the equation of motion: the solitons. This solution can be obtained through the variational principle in the following way:

$$-\delta\mathcal{L} = \delta\int [\frac{1}{2}(\partial_x\phi)^2 + V(\phi)]dx \tag{3}$$

This is mathematically equivalent to the problem of a particle with unit mass in a potential $-V(x)$. The equation of motion comes from minimizing the action:

$$\delta\mathcal{L}dt = \delta\int [\frac{1}{2}(\frac{dx}{dt})^2 + V(x)]dt = 0 \tag{4}$$

The motion of the particle in the potential $-V(x)$ is analogous to a time independent field solution. However, to have a finite-energy solution, the field has to go to the zero of $V(\phi)$ as $x \to \pm\infty$, so the following integral is finite:

$$H = \int [\frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 + V(\phi)]dx \tag{5}$$

In the particle - equivalent problem, this means that the particle goes to the zeros of the potential as $t \to \pm\infty$ [5]. So, the solution takes the vacuum values $x = \pm a$ when $t \to \pm\infty$. It can move from one vacuum to another one at different points. We will find now the form of the finite-energy solution starting with the case of zero-energy:

$$\frac{1}{2}(\frac{dx}{dt})^2 - V(x) = 0, \tag{6}$$

or, for if we are studying fields:

$$\frac{1}{2}(\frac{d\phi}{dx})^2 - V(\phi) = 0, \tag{7}$$

Integrating this result, we have:

$$x = \pm \int_{\phi_0}^{\phi} d\phi (2V(\phi))^{-\frac{1}{2}}, \tag{8}$$

where $-a < \phi_0 < a$.

The solutions are invariant under translations. If $\phi = f(x)$ is a solution, then $\phi = f(x + a)$ is also a solution, where a is a constant.

For the case of $\lambda\phi^4$ theory, whith a potential given by equation (2), the finite-energy solutions given by equation (8) are:

$$\phi_+(x) = atanh(\mu x) \tag{9}$$

$$\phi_-(x) = -atanh(\mu x) \tag{10}$$

Solutions (9) and (10) are know as *kink* and *anti-kink*, respectively. Their energy is finite, and it is expressed as

$$E = \frac{4\mu^3}{3\lambda} \tag{11}$$

It is interesting to note that if we consider the Euclidean version of the field theory, indentify the time coordinate as the y coordinate, and think of $\phi(x, y)$ as the magnetization, this configuration describes a domain wall in a two-dimensional magnetic system (see Figure 1).

From (9), we can see that $\phi_+ = \pm a$ as $x \to \pm\infty$. Even though these solutions are not the absolute minimum of the potential, they are stable.

This finite-energy solutions resemble a particle in the following way:

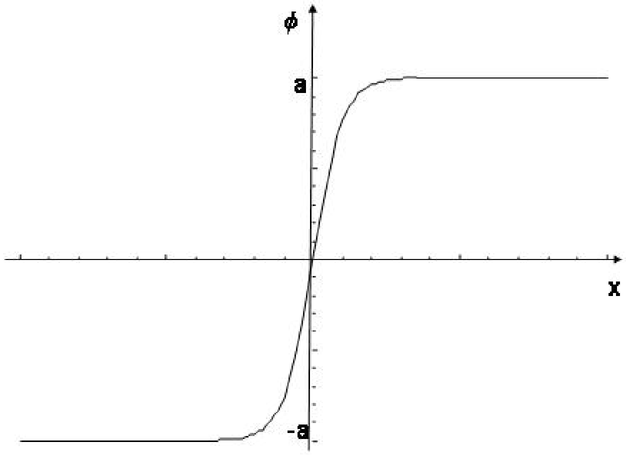


Figure 1: Configuration in $1 + 1$ dimensions.

- The energy is concentrated in a finite region of space. This is due to the fact that $\phi\pm$ deviate from the ground state $\phi = \pm a$ only in a small region.

- It can be made to move with any velocity less than unity. The equation is Lorentz-covariant, so we can make a Lorentz boost to obtain a solution with non-zero velocity.

# 2 More dimensions: Derrick theorem.

**Derrick theorem.**

If $\phi$ is a set of scalar field in one time dimension and D space dimensions, with a potential $U(x) \geq 0$, and $U(x) = 0$ for the ground state of the theory, then, for $D \geq 2$, the only non-singular time-independent solutions of finite energy are the ground state [2].

This is sufficiently discouriging to make us investigate theories of more than 2 spatial dimensions.

**Proof**

Define

$$V_1 = \frac{1}{2}\int d^D x (\nabla\phi)^2 \tag{12}$$

and

$$V_2 = \frac{1}{2}\int d^D x U(\phi) \tag{13}$$

$V_1$ and $V_2$ are both non-negative and are simultaneously equal to zero only for the ground states. Let $\phi(\mathbf{x})$ be a time-independent solution. Considering the one-parameter family of field configurations defined by

$$\phi(\mathbf{x};\lambda) \equiv \phi(\lambda\mathbf{x}), \tag{14}$$

the energy is given by

$$V(\lambda) = \lambda^{(2-D)} V_1 + \lambda^{-D} V_2 \tag{15}$$

This must be stationary at $\lambda = 1$ (by Hamilton s principle). Then,

$$(2-D)V_1 + DV_2 = 0 \tag{16}$$

For $D > 2$, this implies that both $V_1$ and $V_2$ vanish, and the proof is complete. For $D = 2$, this only implies that $V_2$ vanishes. $V_2$ is also stationary since zero is its minimum value. So, applying Hamiltons principle to $V_1$ alone, it trivially follows that $V_1$ also vanishes.

# 3 The mass of the kink. A non perturbative result.

The mass of the kink can be calculated by minimizing:

$$M = \int dx [\frac{1}{2}(\frac{d\phi}{dx})^2 + \frac{\lambda}{4}(\phi^2 - v^2)^2] \tag{17}$$

Scaling $\phi(x) \to v f(y)$, and $y = \mu x$, we obtain:

$$M = \frac{\mu^2}{\lambda}\mu \int dy [\frac{1}{2}(\frac{df}{dy})^2 + \frac{1}{4}(f^2 - 1)^2] \tag{18}$$

The mass of the kink results to be $M = \frac{2\sqrt{2}}{3}\frac{\mu^3}{\lambda} + \mu(\frac{1}{6}\sqrt{\frac{3}{2}} - \frac{3}{\pi\sqrt{2}})$, calculated by Dashen et al. in 1974.

We can make the following observations about this expression: (i) The first term in the mass of the quantum kink particle is the energy of the classical static kink solution. The next term represents the correction due to quantum fluctuations.

(ii) The first term - the energy of the classical kink - is singular as $\lambda \to 0$. So, this result is non perturbative. It could not have been obtained from a perturbation expansion starting from the vacuum.

(iii) Nevertheless, the quantum fluctuations are being treated perturbatively in powers of $\lambda$. This result is only valid in the weak coupling approximation.

Another way of constraining the mass is known as the Bogomolnyi inequality (we will discuss it in the following subsection).

### 3.1 Bogomolnyi inequality.

The energy density M (equation 17) is the sum of two squares. Using the property $a^2 + b^2 \geq 2\,|a.b|$, we get

$$M \geq \int dx (\frac{\lambda}{2})^{\frac{1}{2}} \left| \frac{d\phi}{dx}(\phi^2 - v^2) \right| \geq \left| \frac{4}{3\sqrt{2}} \mu (\frac{\mu^2}{\lambda}) Q \right|, \tag{19}$$

where Q is defined below (equation 23)

Then, $M \geq |Q|$, and we have constrained the mass of the kink [2].

# 4 Topological properties

The *kink* and *anti-kink* solutions in $1 + 1$ dimensions have interesting topological properties. These topological properties make the solutions stable.

In $1 + 1$ dimensions, the $\lambda\phi^4$ solutions satisfy the following properties:

$$\phi(\infty) - \phi(-\infty) = n(2a), \tag{20}$$

where $n = 0$ corresponds to the ground state, $n = 1$ to the *kink* solution, and $n = -1$ to the *anti-kink* solution. This can also be written as

$$\int_{-\infty}^{\infty} (\partial_x \phi) dx = n(2a) \tag{21}$$

Then, if we define a current $j_\mu$ as

$$j_\mu(x) = \epsilon_{\mu\nu} \partial^\nu \phi, \tag{22}$$

the current will be conserved: $\partial_\mu j^\mu = 0$ (the tensor $\epsilon^{\mu\nu}$ is an antisymmetric tensor), and the conserved charge is given by

$$Q = \int_{-\infty}^{\infty} j_0 dx = \int_{-\infty}^{\infty} (\partial_x \phi) dx = n(2a) \tag{23}$$

Therefore, the kink number n is a conserved quantum number and, consequently, there is no possible transition between kink or anti-kink solutions and the ground state: they are stable. This is usually known as the *topological conservation law* [1].

The current $j^\mu$ is known as a topological current, and its existence does not follow from Noether s theorem but from topology. Another way of seeing this is that one would need an infinite amount of energy in order to change the value of $\phi$ from $-a$ to $a$ for x from some point to infinity, for example. Small localized packets of oscillations in the field (mesons) clearly have $Q = 0$, while the kink has $Q = 1$. Then, the kink cannot decay into a bunch of mesons.

The different sectors (with different values of n) can be characterized by their topological properties as follows. As we said before, as $x \to \pm\infty$, $\phi_+$ or $\phi_-$ approaches the zeros of $V(\phi)$. We will denote the set of spatial infinities of the theory (the two discrete points $+\infty$ and $-\infty$) by S and the set of minima of the potential $\pm a$ by $M_0$. The condition that the solution to the equation of motion has finite energy implies that the asymptotic values of $\phi(x)$ are zeros of $V(\phi)$:

$$lim_{x\to\pm\infty}\phi(x) = \phi \in M_0 \tag{24}$$

This can be considered as a mapping from points in S to $M_0$. In the ground state configuration, $\pm\infty$ are mapped to $a$, and in the kink configuration $\phi_+$ maps $+\infty$ to $+a$ and $-\infty$ to $-a$. These are topologically distinct mappings (one cannot deform one into another).

# 5 Solitons in $3 + 1$ dimensions.

In $3 + 1$ dimensions, in order to have topologically stable finite-energy solutions, we must have long-range magnetic fields. The Lagrangian density for a $3 + 1$ dimensional theory with a finite-energy solution for a scalar field can be written as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 - V(\phi_i) \tag{25}$$

We will denote again with $M_0$ the set of minima of the potential:

$$M_0 = \phi_i = \mu_i, V(\mu_i = 0)$$

If the theory has a symmetry G, the elements of $M_0$ will be related by the symmetry group G. As an example, $\phi = \pm a$ in $\lambda\phi^4$ theory are related by the symmetry $\phi = -\phi$. The possible directions in which $\mathbf{r}$ can go to infinity are given by a unit vector defined in a sphere in three dimensional space ( a two-sphere):

$$S^2 = \hat{r}, \hat{r}^2 = 1$$

We note that, in four dimensions, $S^2$ is a connected set.

The condition for a finite-energy solution is that as $r \to \infty$, $\phi$ approaches one of the zeros of $V(\phi)$:

$$\phi_i^\infty(\hat{r}) \in M_0$$

For the ground state configuration, $\phi_i^\infty$ goes to the same value in all directions. If G is a discrete symmetry group, $M_0$ is a discrete set, and as $S^2$ is connected, $\phi_i^\infty$ would have to be constant. Then, $\phi_i^\infty$ will have the same topology as the vacuum configuration, and is topologically trivial. To have a topologically non-trivial solution, the symmetry G has to be continuous. This non-trivial topological solution correspond to a long-range magnetic field. To study this, we first notice that the energy is bounded by:

$$H \geq \int d^3x [\frac{1}{2}(\nabla\phi_i)^2 + V(\phi_i)], \tag{26}$$

where

$$(\nabla\phi)^2 = (\frac{\partial\phi}{\partial r})^2 + (\hat{r} \times \nabla\phi)^2 \tag{27}$$

Since $\phi_i^\infty$ is a constant, $(\nabla\phi)^2$ will go like $r^{-2}$ as $r \to \infty$, and therefore the integral in H will be divergent. In conclusion, there are no topologically stable finite-energy solutions in $3+1$ dimensions.

If we add gauge fields to the theory, we can deal with this difficulty. So, we will replace the gradient by a covariant derivative:

$$D_i\phi = \nabla_i\phi + ig(A_i^a T_a)\phi \tag{28}$$

This way, it is possible to have $D_i\phi$ decreasing as $r^{-2}$ and, therefore, a convergent energy integral for non-trivial topological solutions. In this case, the gauge field $A_i^a$ decreases as $r^{-1}$, and the field strength as $r^{-2}$, which corresponds to a long-range magnetic field.

# 6 Homotopy groups

Spacial infinity is topologically a unit circle $S^1$ in a two dimensional space, and the field configuration where $\phi = \pm a$ also forms a circle $S^1$. So, this can be characterized as a map $S^1 \to S^1$. Since this map cannot be deformed into a map in which $S^1$ is mapped into a point in $S^1$, the field configuration is topologically stable. The homotopy group $\Pi_n(M)$ classifies maps of $S_n$

into a manifold M counting the number of topologically inequivalent maps. With this language, we can discuss topological solitons. In the case of solitons, we can see that a kink is a physical manifestation of $\Pi_0(S^0) = Z_2$. Here, the zero dimensional sphere $S^0$ consists of two points, $(-a; a)$, and is topologically equivalent to the spatial infinity in one dimension [4].

# 7 Dynamically generated kinks

As I mentioned above, the existence of kinks and solitons follows from general considerations of symmetry and topology, rather than from dynamics. If we have a have a $1+1$ dimensional theory with a discrete G symmetry, we will expect a kink (a time independent independent configuration $\sigma$(x)), such that

$$\sigma(-\infty) = -\sigma_{min} \tag{29}$$

and

$$\sigma(+\infty) = \sigma_{min} \tag{30}$$

(The anti-kink has $\sigma(-\infty) = \sigma_{min}$ and $\sigma(+\infty) = -\sigma_{min}$).

Studying field theories in the large N limit, the action of a fermionic theory is

$$S(\psi, \sigma) = \int dx[\sum_{a=1}^{N} \overline{\psi}_a(i\partial_\mu\gamma^\mu - \sigma)\psi_a - \frac{N}{2g^2}\sigma^2], \tag{31}$$

where the scalar field $\sigma_{min}$ can be seen as the mass acquired by a fermion.

Integration over the fermionic fields leaves:

$$S(\sigma) = -\int dx \frac{N}{2g^2}\sigma^2 - iNtr(log((i\partial_\mu\gamma^\mu - \sigma))) \tag{32}$$

The factor of N is counting for the integration over N fermionic fields. To find the precise shape of the kink , one would have to evaluate the trace in equation (32), for an arbitrary function $\sigma(x)$ , such that $\sigma(\infty) = -\sigma(-\infty)$ and then, varying the functional of $\sigma(x)$, it would be possible to find the optimal shape of the kink.

We can make the calculation of the trace in the following way:

$$tr(log(i\partial_\mu\gamma^\mu - \sigma(x))) = tr(log\gamma^5(i\partial_\mu\gamma^\mu - \sigma(x))\gamma^5) = \tag{33}$$

$$= tr(log((-1)(i\partial_\mu\gamma^\mu + \sigma(x)))) = \frac{1}{2}tr(log((i\partial_\mu\gamma^\mu - \sigma(x))(i\partial_\mu\gamma^\mu + \sigma(x)))) =$$

$$= \frac{1}{2}tr(log([-\partial^2 + i\gamma^1\sigma(x) - (\sigma(x))^2]))$$

$\gamma^1$ has eigenvalues equal to $\pm i$. Therefore, we reach the following result:

$$tr(log((i\partial_\mu\gamma^\mu - \sigma(x)))) = tr(log([-\partial^2 - \sigma(x) - (\sigma(x))^2])) \tag{34}$$

So, the action defined in equation (32) has a term quadratic in $\sigma(x)$ and a term that depends on the combination $[\sigma(x) + (\sigma(x))^2]$.

The solution for the soliton ($\sigma_{min}$) minimizes the action. Then, the soliton is given by the solution of this differential equation:

$$\sigma(x) + (\sigma(x))^2 = \sigma_{min}^2 \tag{35}$$

This has the following result:

$$\sigma(x) = \sigma_{min} tanh(\sigma_{min} x),$$

and the size of the soliton can be determined as $\frac{1}{\sigma_{min}} = \frac{1}{m_{ferm}}$

# 8 Soliton - soliton interaction

Guided by the analogy described in section 1 between solitons and particles, we want to see if the dynamics of a set of two or more solitons can be described by an interaction potential which depends on their relative separation.

Since the field equations are non-linear, a superposition of single-soliton functions will not be a solution, in general. But if the solitons are far enough, the non-linear effects will cause a small distortion in each soliton, and the overlap will be small. In general, these solutions will not be static because each soliton will exert some force on the others, and make them accelerate.

As an example, we will consider a $\lambda\phi^4$ model. Boundary conditions permit kink-antikink configurations.

The time independent field equation is given by:

$$\phi'' - \lambda\phi^3 + m^2\phi = 0 \tag{36}$$

The only finite-energy solutions of this equation are the kink, the anti-kink and the trivial solution $\phi(x) = \pm\frac{m}{\sqrt{\lambda}}$. The kink and anti-kink solutions will exert force on one another and will not remain static. But they could remain stationary by applying some external force. This holding can be done in the following way [3].

We will consider the modified equation of motion:

$$\phi'' - \lambda\phi^3 + m^2\phi = \alpha(R)[\delta(x - \frac{R}{2}) + \delta(x - \frac{R}{2})] \tag{37}$$

The right hand side can be viewed as two external point forces applied at $x = \pm\frac{R}{2}$.

This solution does, in fact, yield to a finite-energy solution that looks like a kink - anti kink pair separated by a distance R. We note that this solution will have a slope discontinuity at the points $x = \pm\frac{R}{2}$. The distortion goes to zero as $R \to \infty$. The solutions are elliptic integrals (Rajaraman, 1977). The energy corresponding to this solution is

$$E(R) = \frac{4\sqrt{2}m^3}{3\lambda} - 8\sqrt{2}\frac{m^3}{\lambda}exp(-\sqrt{2}mR) + \mathcal{O}(R) \tag{38}$$

The first term is the sum of the masses of the free kink and anti-kink. The second term has the form of an interaction potential energy V(R) of the kink and anti-kink pair:

$$V(R) \;\; -8\sqrt{2}\frac{m^3}{\lambda}exp(-\sqrt{2}mR)$$

This interaction is attractive and strong when the coupling $\frac{\lambda}{m^3}$ is weak and it is valid only for large R, since at small R the kink and anti-kink loose their identity.

We therefore strengthened the analogy of kink and anti-kink solutions with Newtonian particles.

# References


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